LOW RUN RESEARCH CENTER
ING RESEARCH INSTITUTE ~ UNIVERSITY OF MICHIGAN

Studies in Radar Cross-Sections - IX

Electromagnetic Scattering by an Oblate Spheroid

by L. M. Rauch

Project MIRO
Contract No. AF 30(602)-9
UMM-116 October, 1953

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STUDIES IN RADAR CROSS-SECTIONS

I Scattering by a Prolate Spheroid by F. V. Schultz (March 1950).

II The Zeros of the Associated Legendre Functions $P_n^m(\mu')$ of Non-Integral Degree by K. M. Siegel, D. M. Brown, H. E. Hunter, H. A. Alperin, and C. W. Quillen (April 1951).

III Scattering by a Cone by K. M. Siegel and H. A. Alperin (January 1952).

IV Comparison Between Theory and Experiment of the Cross-Section of a Cone by K. M. Siegel, H. A. Alperin, J. W. Crispin, H. E. Hunter, R. E. Kleinman, W. C. Orthwein, and C. E. Schensted (February 1953).

V A Classified Paper on Bistatic Radars by K. M. Siegel (August 1952).

VI Cross-Sections of Corner Reflectors and Other Multiple Scatterers at Microwave Frequencies by R. R. Bonkowski, C. R. Lubitz, and C. E. Schensted (October 1953).

VII A Classified Summary Report by K. M. Siegel, J. W. Crispin, and R. E. Kleinman (November 1952).

VIII Theoretical Cross-Sections as a Function of Separation Angle Between Transmitter and Receiver at Small Wavelengths by K. M. Siegel, H. A. Alperin, R. R. Bonkowski, J. W. Crispin, A. L. Maffett, C. E. Schensted, and I. V. Schensted (October 1953).

IX Electromagnetic Scattering by an Oblate Spheroid by L. M. Rauch (October 1953).

X The Radar Cross-Section of a Sphere by H. Weil (To be published).

XI The Numerical Determination of the Radar Cross-Section of a Prolate Spheroid by K. M. Siegel, B. H. Gere, I. Marx, and F. B. Sleator (December 1953).

XII A Classified Summary Report by K. M. Siegel, M. E. Anderson, R. R. Bonkowski, and W. C. Orthwein (December 1953).

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PREFACE

This paper is the ninth in a series of reports growing out of studies of radar cross-sections at the University of Michigan's Willow Run Research Center. The primary aims of this program are:

1. To show that radar cross-sections can be determined analytically.
2. To elaborate means for computing cross-sections of objects of military interest.
3. To demonstrate that these theoretical cross-sections are in agreement with experimentally determined values.

Intermediate objectives are:

1. To compute the exact theoretical cross-sections of various simple bodies by solution of the appropriate boundary-value problems arising from the electromagnetic vector wave equation.
2. To examine the various approximations possible in this problem, and determine the limits of their validity and utility.
3. To find means of combining the simple-body solutions in order to determine the cross-sections of composite bodies.
4. To tabulate various formulas and functions necessary to enable such computations to be done quickly for arbitrary objects.
5. To collect, summarize, and evaluate existing experimental data.

Titles of the papers already published or presently in process of publication are listed on the back of the title page.

K. M. Siegel

NOMENCLATURE

A	= semi-major axis of the oblate spheroid.
$A_{\ell 0}$	= a condensation symbol -- Equation (44).
B	= semi-minor axis of the oblate spheroid.
$B_{\ell}^{L}, D_{\ell}^{L}, R_{\ell}^{L}, S_{\ell}^{L}, T_{\ell}^{L}$	= condensation symbols -- Equation (77).
$B(t, k)$	= Beta function.
$C_{2k}^{\ell m}$	= the oblate spheroidal coefficient.
$\underline{E}^{I}$	= incident electric field vector.
$\underline{E}_x^{I}, \underline{E}_y^{I}, \underline{E}_z^{I}$	= x, y, z components of the incident electric field vector.
$\underline{E}_0$	= the incident electric vector $\underline{E}^{I}$ with the phase factor removed.
E_0	= magnitude of the incident field vector $\underline{E}^{I}$.
E, O	= denotes whether ℓ and L are even or odd in the $I_n(\ell, L)$.
F	= 1/2 the focal length of the coordinate ellipsoid and hyperboloid.
$\underline{H}^{I}$	= incident magnetic field vector.
$\underline{H}_x^{I}, \underline{H}_y^{I}, \underline{H}_z^{I}$	= x, y, z components of the incident magnetic field vector $\underline{H}^{I}$.
H_0	= magnitude of the incident magnetic field vector $\underline{H}^{I}$.

NOMENCLATURE (Continued)

$\underline{H}_0$	= the incident magnetic field vector $\underline{H}^I$, with the phase factor removed.
$I_n(\ell, L)$	= integrals occurring in Equations (68) and (69).
$(\ )\underline{M}^a_{\ell m}$	= a solution to the vector wave equation constructed from solution to the scalar Helmholtz equation.
$N_{\ell m}$	= the norms of the angular functions.
P	= a point of observation in space given in either oblate spheroidal coordinates (η, ξ, ϕ) or spherical coordinates (r, θ, ϕ).
P^I	= total power intercepted by an isotropic scatterer of cross-section σ.
$\underline{P}^I$	= Poynting vector of incident electromagnetic field.
$\underline{P}^S$	= Poynting vector of scattered electromagnetic field.
Q	= the location of a radiating source in space given in either oblate spheroidal coordinates (η', ξ', ϕ') or spherical coordinates (r', θ', ϕ').
$R(\eta), S(\xi), \Phi(\phi)$	= solutions of the separated scalar Helmholtz equation.
$U_{\ell m}(\eta)$	= angular oblate spheroidal wave function.
$V_{\ell m}(\xi)$	= radial oblate spheroidal wave function.

NOMENCLATURE (Continued)

a	= an index used to indicate any of the x, y, or z coordinates.
$\underline{a}$	= a vector used to indicate any of the unit vectors $\underline{i}_x$, $\underline{i}_y$, or $\underline{i}_z$.
$f_{2k}^{\ell m}$	= the coefficients in the power series expansion of $\gamma_{\ell m}$.
i	= the imaginary unit $\sqrt{-1}$.
$\underline{i}_x, \underline{i}_y, \underline{i}_z$	= unit vectors in rectangular Cartesian coordinates.
$\underline{i}_\eta, \underline{i}_\xi, \underline{i}_\phi$	= unit vectors in oblate spheroidal coordinate system.
ℓ	= an integer.
k	= $2\pi/\lambda$.
m	= a separation constant which is found to be an integer.
$n_\nu(\epsilon, \xi)$	= the spherical Neumann function.
$q_{\ell m}$	= the coefficients in the expansion of the radial functions of the second kind.
r	= the radial distance from the origin to a point in space.
∇	= the differential operator $\underline{i}_x \frac{\partial}{\partial x} + \underline{i}_y \frac{\partial}{\partial y} + \underline{i}_z \frac{\partial}{\partial z}$.
α_ℓ, β_ℓ	= arbitrary constant determined by boundary conditions.

NOMENCLATURE (Continued)

$\alpha_{\ell m}$ = a separation constant.

$\gamma_{\ell m}$ = a separation constant.

$\delta_{\ell m}$ = the Kronecker delta.

ϵ = kF.

η = coordinate hyperboloid of the oblate spheroidal coordinate system.

λ = wavelength.

ξ = coordinate ellipsoid of the oblate spheroidal coordinate system.

ρ = distance between two points P and Q in space.

σ = scattering cross-section.

ϕ = coordinate angle of the oblate spheroidal coordinate system measured counter-clockwise around the z-axis from the xz-plane.

ψ = a solution of the scalar Helmholtz equation.

I

INTRODUCTION AND SUMMARY

At the present time, exact solutions are known for the electromagnetic scattering cross-sections of only four three-dimensional configurations: the sphere, the paraboloid, the semi-infinite cone, and the prolate spheroid. In this paper, the exact back-scattering cross-section of a fifth three-dimensional configuration, the oblate spheroid, is obtained. Although this body may be of little practical importance, the scattering cross-section has been obtained as part of the first intermediate objective stated in the Preface, to widen the theoretical knowledge of this field. The solution is obtained for one direction of incidence and presented in the form of an infinite series. Much work still remains to be done in developing methods for evaluating this series to obtain numerical values of cross-section.

The electromagnetic scattering cross-section of the oblate spheroid has been obtained as follows: the oblate spheroidal coordinate system, in which the scalar wave equation is separable and which includes the scattering surface as a coordinate surface, is chosen. A plane, linearly polarized wave progressing along the z-axis in the negative z-direction is incident on the surface of a perfectly conducting oblate spheroid. Using the method of Hansen in which a solution of the vector wave equation is constructed from the solution of a scalar Helmholtz equation, an expression is obtained for the scattered electric field in series form having a number of arbitrary constants. This expression is valid everywhere in space. By means of the boundary conditions at the surface of the scatterer, the properties of the scattered radiation are related to those of the incident radiation so as to obtain defining relations for the arbitrary constants. Unfortunately, these defining relations are quite complex and no simple technique has yet been developed for determining the exact values of the required constants.

The scattered electric field is used to determine the back-scattering cross-section of the oblate spheroid. The back-scattering

cross-section, σ, is defined by Ridenour (Ref. 1) as: "σ (dimensions of an area) is to be 4π times the ratio of the power per unit solid angle scattered back toward the transmitter to the power density (power per unit area) in the wave incident on the target. In other words, if at the target the power incident on an area σ placed normal to the beam were to be scattered uniformly in all directions, the intensity of the signal received back at the radar set would be just what it is in the case of the actual target." The expression obtained for the scattering cross-section depends upon the set of oblate spheroidal coefficients which have not been tabulated extensively and upon the above-mentioned arbitrary constants. Therefore, the usefulness of the expression for the cross-section, as far as numerical calculations are concerned, is limited at the present time.

In conclusion, the author wishes to acknowledge his indebtedness to the work of F. V. Schultz (Ref. 2) and of A. Leitner and R. D. Spence (Ref. 3).

II

THE OBLATE SPHEROIDAL FUNCTIONS

To obtain expressions for the incident and scattered electromagnetic field and hence the back-scattering cross-section, the properties of the solutions of the vector and scalar wave equations for the spheroid are required. These properties are presented below.

2.1 THE COORDINATE SYSTEM

The solutions of the vector wave equation and of the corresponding scalar wave equation can be expressed in oblate spheroidal coordinates η, ξ, and ϕ. This system consists of three families of orthogonal surfaces which are described by η, ξ, and ϕ equal to constants. The first two of these families consist of confocal surfaces of revolution about the z-axis, namely, hyperboloids of one sheet and oblate spheroids as indicated in Figure 1. The equation ϕ equals a constant represents the family of half planes through the z-axis, where ϕ is the angle between these planes and the xz-plane.

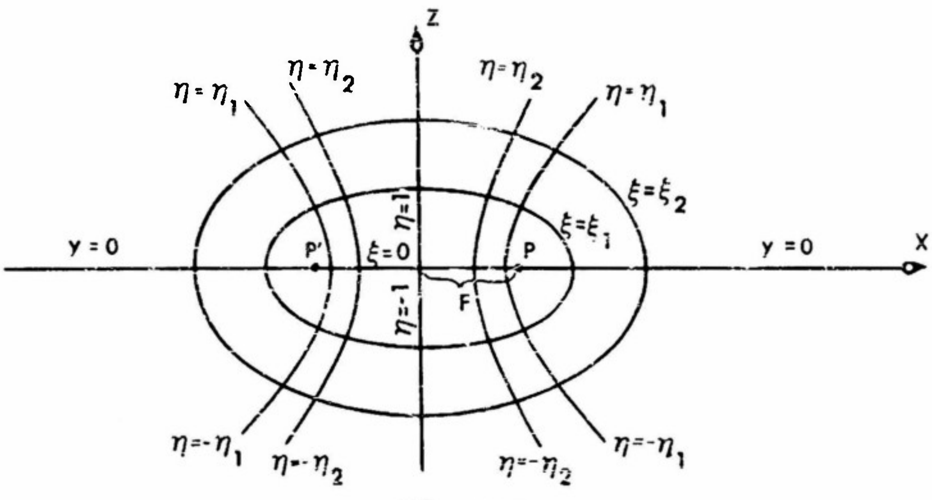

Figure 1

Oblate Spheroidal Coordinates

The equations of transformation from the oblate spheroidal coordinate systems to the Cartesian system are:

$$x = F\left[(1 - \eta^2)(1 + \xi^2)\right]^{1/2} \cos\phi,$$

$$y = F\left[(1 - \eta^2)(1 + \xi^2)\right]^{1/2} \sin\phi,$$

$$z = F\eta\xi,$$

where F is 1/2 the focal length (PP'), $-1 \leq \eta \leq 1$, $0 \leq \xi < \infty$, and $0 \leq \phi < 2\pi$. In the Cartesian system the equation of the oblate spheroid $\xi = \xi_0$ is

$$\frac{x^2 + y^2}{A^2} + \frac{z^2}{B^2} = 1,$$

where A and B are the semi-major and semi-minor axes, respectively, and are given by

$$A = F\sqrt{1 + \xi_0^2}, \quad B = F\xi_0.$$

The distance from the origin to any point in space is specified by

$$r = \sqrt{x^2 + y^2 + z^2} = F\sqrt{1 + \xi^2 - \eta^2}. \tag{1}$$

When ξ is large, r is approximately $F\xi$.

The relations between unit vectors in the Cartesian system and the oblate spheroidal system are

$$\mathbf{i}_x = \mathbf{i}_\eta\left[-\eta(1 + \xi^2)^{1/2}(\xi^2 + \eta^2)^{-1/2}\right]\cos\phi$$

$$+ \mathbf{i}_\xi\left[\xi(1 - \eta^2)^{1/2}(\eta^2 + \xi^2)^{-1/2}\right]\cos\phi - \mathbf{i}_\phi \sin\phi \tag{2a}$$

$$\mathbf{i}_y = \mathbf{i}_\eta\left[-\eta(1 + \xi^2)^{1/2}(\eta^2 + \xi^2)^{-1/2}\right]\sin\phi$$

$$+ \mathbf{i}_\xi\left[\xi(1 - \eta^2)^{1/2}(\eta^2 + \xi^2)^{-1/2}\right]\sin\phi + \mathbf{i}_\phi \cos\phi \tag{2b}$$

$$\underline{i}_z = \underline{i}_\eta \left[\xi(1-\eta^2)^{1/2}(\eta^2+\xi^2)^{-1/2}\right] + \underline{i}_\xi \left[\eta(1+\xi^2)^{1/2}(\eta^2+\xi^2)^{-1/2}\right] \tag{2c}$$

and

$$\underline{i}_\eta = \underline{i}_x \left[-\eta(1+\xi^2)^{1/2}(\eta^2+\xi^2)^{-1/2}\cos\phi\right] + \underline{i}_y \left[-\eta(1+\xi^2)^{1/2}(\eta^2+\xi^2)^{-1/2}\sin\phi\right] + \underline{i}_z \left[\xi(1-\eta^2)^{1/2}(\eta^2+\xi^2)^{-1/2}\right] \tag{3a}$$

$$\underline{i}_\xi = \underline{i}_x \left[\xi(1-\eta^2)^{1/2}(\eta^2+\xi^2)^{-1/2}\cos\phi\right] + \underline{i}_y \left[\xi(1-\eta^2)^{1/2}(\eta^2+\xi^2)^{-1/2}\sin\phi\right] + \underline{i}_z \left[\eta(1+\xi^2)^{1/2}(\eta^2+\xi^2)^{-1/2}\right] \tag{3b}$$

$$\underline{i}_\phi = -\underline{i}_x \sin\phi + \underline{i}_y \cos\phi . \tag{3c}$$

2.2 THE SOLUTIONS OF THE SCALAR AND VECTOR WAVE EQUATIONS

The scalar Helmholtz equation,

$$\nabla^2 \psi + k^2 \psi = 0, \tag{4}$$

in oblate spheroidal coordinates is

$$\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right] + \frac{\partial}{\partial\xi}\left[(1+\xi^2)\frac{\partial\psi}{\partial\xi}\right] + \left(\frac{1}{1-\eta^2} - \frac{1}{1+\xi^2}\right)\frac{\partial^2\psi}{\partial\phi^2} + \epsilon^2(\eta^2+\xi^2)\psi = 0 \tag{5}$$

where $\epsilon = kF = \frac{2\pi}{\lambda}F$, $k = \frac{\epsilon\xi_0}{B}$, and λ denotes wavelength.

Equation (5) can be separated by assuming a solution of the form

$$\psi(\eta, \xi, \phi) = R(\eta)\, S(\xi)\, \Phi(\phi). \tag{6}$$

The substitution of Equation (6) into (5) yields

$$\frac{d}{d\eta}\left[(1-\eta^2)\cdot\frac{dR(\eta)}{d\eta}\right]-\left[\frac{m^2}{1-\eta^2}-a+\epsilon^2(1-\eta^2)\right]R(\eta)=0 \qquad (7)$$

$$\frac{d}{d\xi}\left[(1+\xi^2)\cdot\frac{dS(\xi)}{d\xi}\right]+\left[\frac{m^2}{1+\xi^2}-a+\epsilon^2(1+\xi^2)\right]S(\xi)=0 \qquad (8)$$

$$\frac{d^2\Phi(\phi)}{d\phi^2}+m^2\Phi(\phi)=0 \qquad (9)$$

where m and a are introduced as constants of separation. Since it will be convenient later to express a in terms of m and another constant, ℓ, a is written as $a_{\ell m}$. Particular solutions of Equations (7) and (8) will be denoted by $R(\eta) = U_{\ell m}(\eta)$ and $S(\xi) = V_{\ell m}(\xi)$. The functions $U_{\ell m}(\eta)$ and $V_{\ell m}(\xi)$ are known as the angular and radial functions respectively. A solution of Equation (9) is $\Phi(\phi) = \cos m\phi$. To insure single valuedness, m is restricted to integral values.

Hansen (Ref. 4) has shown how a solution of the vector wave equation

$$\nabla\nabla\cdot\underline{V}-\nabla\times(\nabla\times\underline{V})+k^2\underline{V}=0 \qquad (10)$$

can be constructed from a solution to the scalar Helmholtz equation by using the solenoidal relation

$$\underline{M}^{(a)}=\nabla\times\underline{a}\,\psi=\nabla\psi\times\underline{a} \qquad (11)$$

where $\underline{a}$ is an arbitrary constant vector and ψ is a solution to the scalar wave equation. Furthermore, the vector $\underline{V}$ can be expressed in terms of $\underline{M}^{(a)}$. In particular, $\underline{a}$ is restricted to the Cartesian unit vectors $\underline{i}_x$, $\underline{i}_y$, and $\underline{i}_z$ and, correspondingly, $\underline{M}^{(a)}$ is restricted to $\underline{M}^x$, $\underline{M}^y$, and $\underline{M}^z$. The function ψ is a Lamé product of the functions $R(\eta)$, $S(\xi)$, and $\Phi(\phi)$.

2.3 EXPRESSIONS FOR THE ANGULAR FUNCTIONS

Leitner and Spence (Ref. 3) have presented a thorough discussion of the properties of the angular functions. Since these properties are used quite extensively here, they are quoted verbatim.

"Equations (7) and (8) may be obtained from the general equation

$$\frac{d}{dz}\left[(1 - z^2)\frac{d F_{\ell m}}{dz}\right] - \left[\frac{m^2}{1 - z^2} - a_{\ell m} + \epsilon^2 (1 - z^2)\right] F_{\ell m} = 0 \qquad (12)$$

by replacing the complex variable z with η or $i\xi$, respectively. Equation (12) differs from the equation of the associated Legendre functions only in that it possesses an irregular singular point at $z = \infty$. The exponents associated with the regular singular points at $z = \pm 1$ are $\pm m/2$. A solution corresponding to the positive exponent will be called a function of the first kind; the other independent solution will be called a function of the second kind. Since the exponent difference is an integer the solutions of the second kind will possess logarithmic singularities at $z = \pm 1$. Since this point corresponds to $\eta = \pm 1$ or $\xi = \pm i$ it lies in the physical range of η but outside the physical range of ξ. Therefore, the angular functions are functions of the first kind while the radial functions are functions of both the first and second kind.

"The analogy between (7) and the equation for the associated Legendre function suggests series of the form

$$U_{\ell m}(\eta) = (1 - \eta^2)^{m/2} \sum_{k=0}^{\infty} C_{2k}^{\ell m} (1 - \eta^2)^k \quad (\ell - m) \text{ even,} \qquad (13)$$

$$U_{\ell m}(\eta) = \eta(1 - \eta^2)^{m/2} \sum_{k=0}^{\infty} C_{2k}^{\ell m} (1 - \eta^2)^k \quad (\ell - m) \text{ odd.} \qquad (14)$$

Also by analogy

$$a_{\ell m} = \ell(\ell + 1) + \gamma_{\ell m} \qquad (15)$$

where $\gamma_{\ell m}$ is a function of ϵ^2. As in the case of the associated Legendre equation, ℓ is restricted to integral values equal to or greater than m. This notation follows that of Page (Ref. 5). Stratton et al. (Ref. 6) use the symbol, ℓ, to denote $(\ell - m)$ as used here. The recursion formulas for the coefficients in (13) and (14) are

$$2k(2k + 2m) C_{2k}^{\ell m} - \left[(2k - \ell + m - 2)(2k + \ell + m - 1) - \gamma_{\ell m}\right] C_{2k-2}^{\ell m} - \epsilon^2 C_{2k-4}^{\ell m} = 0, \quad (\ell - m) \text{ even}, \qquad (16)$$

$$2k(2k + 2m) C_{2k}^{\ell m} - \left[(2k - \ell + m - 1)(2k + \ell + m) - \gamma_{\ell m}\right] C_{2k-2}^{\ell m} - \epsilon^2 C_{2k-4}^{\ell m} = 0, \quad (\ell - m) \text{ odd} \qquad (17)$$

with $C_0^{\ell m} \equiv 1$ in either case.[1] The magnitude of the $C_{2k}^{\ell m}$ begins to decrease when $2k > (\ell - m) + 2$ if $(\ell - m)$ is even and for $2k > (\ell - m) + 1$ if $(\ell - m)$ is odd. For large values of k

$$\frac{C_{2k}^{\ell m}}{C_{2k-2}^{\ell m}} \longrightarrow 1 + \frac{\epsilon^2}{4k^2\left(\dfrac{C_{2k-2}^{\ell m}}{C_{2k-4}^{\ell m}}\right)} \qquad (18)$$

---

[1]These recurrence relations were omitted by Leitner and Spence. Their continued fraction expansions (Eqs. (19) and (20)) have been corrected.

$$4(1 + m) C_2^{\ell m} - \left[(m - \ell)(m + \ell + 1) - \gamma_{\ell m}\right] C_0^{\ell m} = 0, \quad (\ell - m) \text{ even}.$$

$$4(1 + m) C_2^{\ell m} - \left[(m - \ell + 1)(m + \ell + 2) - \gamma_{\ell m}\right] C_0^{\ell m} = 0, \quad (\ell - m) \text{ odd}.$$

regardless of $(\ell - m)$. As k approaches infinity this ratio must either approach unity or approach zero as $-\epsilon^2/4k^2$. In the latter case the $C_{2k}^{\ell m}$ alternate in sign at large k and the series (13) and (14) converge absolutely for all finite η. From (16) and (17) one may easily obtain the continued fractions

$$\gamma_{\ell m} = -\cfrac{(\ell+m)(\ell-m)\epsilon^2}{2(2\ell-1)+\gamma_{\ell m}+\cfrac{(\ell+m-2)(\ell-m-2)\epsilon^2}{4(2\ell-3)+\gamma_{\ell m}+\cfrac{(\ell+m-4)(\ell-m-4)\epsilon^2}{6(2\ell-5)+\gamma_{\ell m}+\cdots}}}$$

$$+\cfrac{(\ell+m+2)(\ell-m+2)\epsilon^2}{2(2\ell+3)-\gamma_{\ell m}+\cfrac{(\ell+m+4)(\ell-m+4)\epsilon^2}{4(2\ell+5)-\gamma_{\ell m}+\cfrac{(\ell+m+6)(\ell-m+6)\epsilon^2}{6(2\ell+7)-\gamma_{\ell m}+\cdots}}}$$

$(\ell-m)$ even. (19)

$$\gamma_{\ell m} = -\cfrac{(\ell+m-1)(\ell-m-1)\epsilon^2}{2(2\ell-1)+\gamma_{\ell m}+\cfrac{(\ell+m-3)(\ell-m-3)\epsilon^2}{4(2\ell-3)+\gamma_{\ell m}+\cfrac{(\ell+m-5)(\ell-m-5)\epsilon^2}{6(2\ell-5)+\gamma_{\ell m}+\cdots}}}$$

$$+\cfrac{(\ell+m+1)(\ell-m+1)\epsilon^2}{2(2\ell+3)-\gamma_{\ell m}+\cfrac{(\ell+m+3)(\ell-m+3)\epsilon^2}{4(2\ell+5)-\gamma_{\ell m}+\cfrac{(\ell+m+5)(\ell-m+5)\epsilon^2}{6(2\ell+7)-\gamma_{\ell m}+\cdots}}}$$

$(\ell-m)$ odd. (20)

In both (19) and (20) the first continued fraction contains only a finite number of terms while the second contains an infinite number of terms. Equations (19) and (20) constitute transcendental equations whose roots are the eigenvalues of $\gamma_{\ell m}$. The eigenvalues can be computed by cutting off the infinitely extending continued fraction and solving the resulting algebraic equation by successive approximations. By retaining more and more terms in the continued fraction one can obtain numerical values for $\gamma_{\ell m}$ to any desired degree of accuracy.[1]

"A series for the $\gamma_{\ell m}$ in powers of ϵ^2 may be obtained (Ref. 5) by substituting

$$\gamma_{\ell m} = \sum_{k=0}^{\infty} f_{2k}^{\ell m} \, \epsilon^{2k} \tag{21}$$

into (19) or (20) and collecting like powers of ϵ^2. One finds

$$f_2^{\ell m} = 2\frac{\ell(\ell+1) + m^2 - 1}{(2\ell - 1)(2\ell + 3)} \tag{22}$$

$$f_4^{\ell m} = \frac{H(\ell - 1)H(\ell)}{2(2\ell - 1)} - \frac{H(\ell + 1)H(\ell + 2)}{2(2\ell + 3)} \tag{23}$$

$$f_6^{\ell m} = \frac{4m^2 - 1}{(2\ell - 1)(2\ell + 3)}\left[\frac{H(\ell - 1)(H(\ell)}{(2\ell - 1)^2(2\ell - 5)} - \frac{H(\ell + 1)\,H(\ell + 2)}{(2\ell + 3)^2(2\ell + 7)}\right] \tag{24}$$

$$f_8^{\ell m} = \frac{H(\ell - 1)H(\ell)}{2(2\ell - 1)}\left[\frac{4(4m^2 - 1)^2}{(2\ell - 5)^2(2\ell - 1)^4(2\ell + 3)^2} - \frac{f_4^{\ell m}}{2(2\ell - 1)} + \frac{H(\ell - 3)H(\ell - 2)}{2\cdot 4(2\ell - 1)(2\ell - 3)}\right]$$

---

[1]"A method for rapidly finding the roots of such continued fractions was independently developed by Bouwkamp (7) and Blanck (8)."

$$
- \frac{H(\ell+1)H(\ell+2)}{2(2\ell+3)}\left[\frac{4(4m^2-1)^2}{(2\ell-1)^2(2\ell+3)^4(2\ell+7)^2} + \frac{f_4^{\ell m}}{2(2\ell+3)} + \frac{H(\ell+3)H(\ell+4)}{2\cdot 4(2\ell+3)(2\ell+5)}\right], \tag{25}
$$

where

$$
H(x) = \frac{x^2 - m^2}{4x^2 - 1}. \tag{26}
$$

The series given in (21) does not appear to converge sufficiently rapidly to be useful if ϵ is greater than 5. For smaller values of ϵ it may be used to obtain a good approximation for $\gamma_{\ell m}$ which may then be improved by use of (19) and (20).

"The angular functions corresponding to a given ϵ are orthogonal in the interval (-1, 1), that is

$$
\int_{-1}^{1} U_{\ell m}(\eta)\, U_{Lm}(\eta)\, d\eta = \delta_{\ell L} N_{\ell m} \tag{27}
$$

where

$$
N_{\ell m} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{2i}^{\ell m} C_{2j}^{\ell m} \frac{2^{m+i+j+1}(m+i+j)!}{1\cdot 3\cdot 5\cdots[2(m+i+j)+1]}, \quad (\ell\text{-m}) \text{ even.} \tag{28}
$$

$$
N_{\ell m} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{2i}^{\ell m} C_{2j}^{\ell m} \frac{2^{m+i+j+1}(m+i+j)!}{1\cdot 3\cdot 5\cdots[2(m+i+j)+3]}, \quad (\ell\text{-m}) \text{ odd.''} \tag{29}
$$

2.4 EXPRESSIONS FOR THE RADIAL FUNCTIONS

Both independent solutions of the radial equation (8) must be incorporated in the general solution. For the functions of the first and second kind Leitner and Spence give

$$ {}^{(1)}V_{\ell m}(\xi) = \frac{(1+\xi^2)^{m/2}}{U_{\ell m}(0)} \sum_{k=0}^{\infty} C_{2k}^{\ell m} (1+\xi^2)^k, \quad (\ell-m) \text{ even,} \tag{30} $$

$$ {}^{(1)}V_{\ell m}(\xi) = \frac{\xi(1+\xi^2)^{m/2}}{\frac{d}{d\eta} U_{\ell m}(0)} \sum_{k=0}^{\infty} C_{2k}^{\ell m} (1+\xi^2)^k, \quad (\ell-m) \text{ odd,} \tag{31} $$

and

$$ {}^{(2)}V_{\ell m}(\xi) = q_{\ell m}(1+\xi^2)^{m/2} \sum_{k=0}^{\infty} 2^{m+k}(m+k)!\, C_{2k}^{\ell m} \frac{n_{m+k}(\epsilon\xi)}{(\epsilon\xi)^{m+k}}, $$

$$ (\ell-m) \text{ even,} \tag{32} $$

$$ {}^{(2)}V_{\ell m}(\xi) = i q_{\ell m}(1+\xi^2)^{m/2} \sum_{k=0}^{\infty} 2^{m+k}(m+k)!\, C_{2k}^{\ell m} \frac{n_{m+k+1}(\epsilon\xi)}{(\epsilon\xi)^{m+k}}, $$

$$ (\ell-m) \text{ odd,} \tag{33} $$

where $n_\nu(\epsilon\xi)$ represents the spherical Neumann function, and the constants $q_{\ell m}$ are given by

$$ q_{\ell m} = \left\{ \sum_{k=0}^{\infty} \frac{2^{m+k}(m+k)!\, C_{2k}^{\ell m}}{1\cdot 3\cdot 5\cdots[2(m+k)+1]} \right\}^{-1} \quad (\ell-m) \text{ even,} \tag{34} $$

$$ q_{\ell m} = \left\{ \sum_{k=0}^{\infty} \frac{2^{m+k}(m+k)!\, C_{2k}^{\ell m}}{1\cdot 3\cdot 5\cdots[2(m+k)+3]} \right\}^{-1} \quad (\ell-m) \text{ odd.} \tag{35} $$

In the problem being considered, the primary interest is in a combination of those solutions which will represent a diverging wave at large distances from the spheroid. Leitner and Spence show that ${}^{(3)}V_{\ell m}(\xi)$, given by ${}^{(3)}V_{\ell m}(\xi) = {}^{(1)}V_{\ell m}(\xi) + i\,{}^{(2)}V_{\ell m}(\xi)$, represents such a diverging wave.

They also give an asymptotic expression for this function which will be very useful:

$$\lim_{\epsilon\xi \to \infty} {}^{(3)}V_{\ell m}(\xi) = q_{\ell m}(2/\epsilon)^m (-i)^{m+1} \frac{e^{i\epsilon\xi}}{\epsilon\xi}, \quad (\ell\text{-m}) \text{ even,} \tag{36a}$$

$$\lim_{\epsilon\xi \to \infty} {}^{(3)}V_{\ell m}(\xi) = q_{\ell m}(2/\epsilon)^m (-i)^{m+2} \frac{e^{i\epsilon\xi}}{\epsilon\xi}, \quad (\ell\text{-m}) \text{ odd,} \tag{36b}$$

$$\lim_{\epsilon\xi \to \infty} \frac{d}{d\xi} {}^{(3)}V_{\ell m}(\xi) = q_{\ell m}(2/\epsilon)^m \, m! \, (-i)^m \frac{e^{i\epsilon\xi}}{\epsilon\xi}, \quad (\ell\text{-m}) \text{ odd or even.} \tag{37}$$

III

THE INCIDENT ELECTROMAGNETIC FIELD

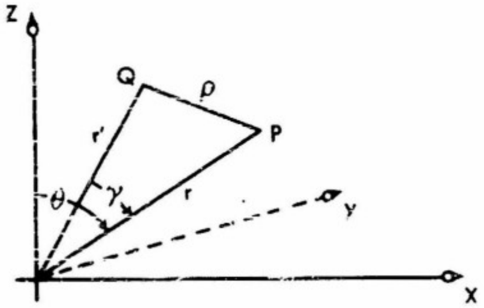

Figure 2
Geometry Used in Describing the Incident Electromagnetic Field.

Let Q be a source point and P a point of observation given by oblate spheroidal and spherical coordinates, P (ξ, η, ϕ, or r, θ, ϕ), Q (ξ', η', ϕ', or r', θ', ϕ'). If ρ is the distance between P and Q and γ the angle between r and r', then

$$\rho = \left[r^2 + (r')^2 - 2r\, r' \cos\gamma \right]^{1/2}, \tag{38}$$

and

$$\cos\gamma = \sin\theta \,\sin\theta' \,\cos(\phi - \phi') + \cos\theta \,\cos\theta'. \tag{39}$$

If the source point Q is moved to infinity along a fixed direction η', ϕ', then for a plane wave progressing along a specified direction (Ref. 3)

$$e^{-ikr\cos\gamma} = \sum_{\ell=0}^{\infty} \left\{ \frac{2(-1)^{\ell}}{N_{\ell 0}\, q_{\ell 0}} U_{\ell 0}(\eta)\, U_{\ell 0}(\eta')\, {}^{(1)}V_{\ell 0}(\xi) + 2 \sum_{m=1}^{\ell} \frac{2^{\ell - m} (-1)^{\ell}\, i^m\, \epsilon^m}{N_{\ell m}\, q_{\ell m}\, m!} \times U_{\ell m}(\eta)\, U_{\ell m}(\eta')\, {}^{(1)}V_{\ell m}(\xi) \cos m(\phi - \phi') \right\} \tag{40}$$

where the $N_{\ell m}$ are given by Equations (28) and (29).

The problem is simplified if the source point is moved to infinity along the positive z-axis in which case $\eta' = +1$, $z = r \cos \gamma$ and Equation (40) becomes

$$e^{-ikz} = \sum_{\ell=0}^{\infty} \frac{2(-1)^{\ell}}{N_{\ell 0}\, q_{\ell 0}} U_{\ell 0}(\eta)\, {}^{(1)}V_{\ell 0}(\xi) \tag{41}$$

where

$$N_{\ell 0} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{2i}^{\ell 0}\, C_{2j}^{\ell 0}\, \frac{2^{i+j+1}(i+j)!}{1\cdot 3 \cdots [2(i+j)+1]}, \quad (\ell-0) \text{ even,} \tag{42}$$

and

$$N_{\ell 0} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{2i}^{\ell 0}\, C_{2j}^{\ell 0}\, \frac{2^{i+j+1}(i+j)!}{1\cdot 3 \cdots [2(i+j)+3]}, \quad (\ell-0) \text{ odd.} \tag{43}$$

Let

$$A_{\ell 0} = \frac{2(-1)^{\ell}}{N_{\ell 0}\, q_{\ell 0}}, \tag{44}$$

and

$${}^{(1)}\psi_{\ell 0}(\eta, \xi) = U_{\ell 0}(\eta)\, {}^{(1)}V_{\ell 0}(\xi), \tag{45}$$

then Equation (41) may be written in the form

$$e^{-ikz} = \sum_{\ell=0}^{\infty} A_{\ell 0}\, {}^{(1)}\psi_{\ell 0}(\eta, \xi). \tag{46}$$

If the incident electric field vector $\underline{E}^{I}$ is chosen so as to have only a positive y-component, then the incident electric and magnetic field vectors ($\underline{E}^{I}$, $\underline{H}^{I}$,) can be expressed as

$$\underline{E}_{y}^{I} = \underline{E}_0\, e^{-ikz} = \underline{i}_y\, E_0 \sum_{\ell=0}^{\infty} A_{\ell 0}\, {}^{(1)}\psi_{\ell 0}(\eta, \xi) \tag{47}$$

and

$$\underline{H}_x^I = \underline{H}_0 e^{-ikz} = \underline{i}_x H_0 \sum_{\ell=0}^{\infty} A_{\ell 0} {}^{(1)}\psi_{\ell 0}(\eta, \xi). \tag{48}$$

Taking the curl of Equations (47) and of (48) yields

$$\nabla \times \underline{E}_y^I = E_0 \sum_{\ell=0}^{\infty} A_{\ell 0} \nabla \times (\underline{i}_y {}^{(1)}\psi_{\ell 0}) \tag{49}$$

and

$$\nabla \times \underline{H}_x^I = H_0 \sum_{\ell=0}^{\infty} A_{\ell 0} \nabla \times (\underline{i}_x {}^{(1)}\psi_{\ell 0}). \tag{50}$$

Assuming a time dependence of the form $e^{-i\omega t}$, Maxwell's equations for free space become

$$\underline{E}_y^I = \frac{i}{k} \nabla \times \underline{H}_x^I \tag{51}$$

and

$$\underline{H}_x^I = \frac{i}{k} \nabla \times \underline{E}_y^I \tag{52}$$

so that $\underline{E}_y^I$ and $\underline{H}_x^I$ are expressible as

$$\underline{E}_y^I = \frac{i}{k} E_0 \sum_{\ell=0}^{\infty} A_{\ell 0} \nabla {}^{(1)}\psi_{\ell 0} \times \underline{i}_x \tag{53}$$

and

$$\underline{H}_x^I = \frac{i}{k} H_0 \sum_{\ell=0}^{\infty} A_{\ell 0} \nabla {}^{(1)}\psi_{\ell 0} \times \underline{i}_y. \tag{54}$$

In terms of a solution $\underline{M}_{\ell m}$ for the vector wave equation, Equations (53) and (54) become

$$\underline{E}_y^I = \frac{i}{k} E_0 \sum_{\ell=0}^{\infty} A_{\ell 0} {}^{(1)}\underline{M}_{\ell 0}^{x} \tag{55}$$

and

$$\underline{H}_x^I = -\frac{i}{k} H_0 \sum_{\ell=0}^{\infty} A_{\ell 0} \, {}^{(1)}\underline{M}_{\ell 0}^y . \tag{56}$$

Using the relations between the unit vectors of the oblate spheroidal coordinate system and the Cartesian system (Eqs. (3a), (3b), and (3c)), the components of the solenoidal vector ${}^{(1)}\underline{M}_{\ell 0}^{(\underline{a})}$ become

$$\begin{aligned}
{}^{(1)}\underline{M}_{\ell 0}^x &= \nabla \, {}^{(1)}\psi_{\ell 0} \times \underline{i}_x \\
&= -\underline{i}_\eta \, F^{-1}\sqrt{1+\xi^2} \, (\eta^2+\xi^2)^{-1/2} \sin\phi \, U_{\ell 0}(\eta) \frac{d}{d\xi} {}^{(1)}V_{\ell 0}(\xi) \\
&\quad + \underline{i}_\xi \, F^{-1}\sqrt{1-\eta^2} \, (\eta^2+\xi^2)^{-1/2} \sin\phi \, {}^{(1)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta) \\
&\quad + \underline{i}_\phi \, F^{-1}\Big[\xi(1-\eta^2)(\eta^2+\xi^2)^{-1} \cos\phi \, {}^{(1)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta) \\
&\qquad + \eta(1+\xi^2)(\eta^2+\xi^2)^{-1} \cos\phi \, U_{\ell 0}(\eta) \frac{d}{d\xi} {}^{(1)}V_{\ell 0}(\xi)\Big]
\end{aligned} \tag{57}$$

and

$$\begin{aligned}
{}^{(1)}\underline{M}_{\ell 0}^y &= \nabla \, {}^{(1)}\psi_{\ell 0} \times \underline{i}_y \\
&= \underline{i}_\eta \, F^{-1}\sqrt{1+\xi^2} \, (\eta^2+\xi^2)^{-1/2} \cos\phi \, U_{\ell 0}(\eta) \frac{d}{d\xi} {}^{(1)}V_{\ell 0}(\xi) \\
&\quad - \underline{i}_\xi \, F^{-1}\sqrt{1-\eta^2} \, (\eta^2+\xi^2)^{-1/2} \cos\phi \, {}^{(1)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta) \\
&\quad + \underline{i}_\phi \, F^{-1}\Big[\xi(1-\eta^2)(\eta^2+\xi^2)^{-1} \sin\phi \, {}^{(1)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta) \\
&\qquad + \eta(1+\xi^2)(\eta^2+\xi^2)^{-1} \sin\phi \, U_{\ell 0}(\eta) \frac{d}{d\xi} {}^{(1)}V_{\ell 0}(\xi)\Big] .
\end{aligned} \tag{58}$$

Hence, the incident electric and magnetic field ($\underline{E}_y^I$, $\underline{H}_x^I$) can now be expressed explicitly as

$$\underline{E}_y^I = \frac{i}{k} E_0 F^{-1} \Bigg\{ \underline{i}_\eta \sum_{\ell=0}^{\infty} A_{\ell 0} \Big[-\sqrt{1+\xi^2}\,(\eta^2+\xi^2)^{-1/2} \sin\phi\, U_{\ell 0}(\eta) \frac{d}{d\xi} {}^{(1)}V_{\ell 0}(\xi) \Big]$$

$$+ \underline{i}_\xi \sum_{\ell=0}^{\infty} A_{\ell 0} \Big[\sqrt{1-\eta^2}\,(\eta^2+\xi^2)^{-1/2} \sin\phi\, {}^{(1)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta) \Big]$$

$$+ \underline{i}_\phi \sum_{\ell=0}^{\infty} A_{\ell 0} \Big[\xi(1-\eta^2)(\eta^2+\xi^2)^{-1} \cos\phi\, {}^{(1)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta)$$

$$+ \eta(1+\xi^2)(\eta^2+\xi^2)^{-1} \cos\phi\, U_{\ell 0}(\eta) \frac{d}{d\xi} {}^{(1)}V_{\ell 0}(\xi) \Big] \Bigg\} \tag{59}$$

and

$$\underline{H}_x^I = -\frac{i}{k} E_0 F^{-1} \Bigg\{ \underline{i}_\eta \sum_{\ell=0}^{\infty} A_{\ell 0} \sqrt{1+\xi^2}\,(\eta^2+\xi^2)^{-1/2} \cos\phi\, U_{\ell 0}(\eta) \frac{d}{d\xi} {}^{(1)}V_{\ell 0}(\xi)$$

$$- \underline{i}_\xi \sum_{\ell=0}^{\infty} A_{\ell 0} \sqrt{1-\eta^2}\,(\eta^2+\xi^2)^{-1/2} \cos\phi\, {}^{(1)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta)$$

$$+ \underline{i}_\phi \sum_{\ell=0}^{\infty} A_{\ell 0} \Big[\xi(1-\eta^2)(\eta^2+\xi^2)^{-1} \sin\phi\, {}^{(1)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta)$$

$$+ \eta(1+\xi^2)(\eta^2+\xi^2)^{-1} \sin\phi\, U_{\ell 0}(\eta) \frac{d}{d\xi} {}^{(1)}V_{\ell 0}(\xi) \Big] \Bigg\}. \tag{60}$$

IV

THE SCATTERED ELECTRIC FIELD

The expression for the scattered electric field must satisfy Maxwell's equations, the boundary conditions at the surface of the oblate spheroid, and the boundary conditions at infinity. In order to satisfy the Maxwell equations it is necessary that the electric field vector satisfy the vector wave equation and the divergence condition. At the surface of the scatterer the tangential component of the total electric field must vanish, and at large distances from the scatterer the scattered electric field must have the form of a spherically divergent wave.

Inspection of the asymptotic form of the radial function of the third kind, ${}^{(3)}V_{\ell m}$, shows that this combination of ${}^{(1)}V_{\ell m}$ and ${}^{(2)}V_{\ell m}$ meets the diverging wave condition. To satisfy the boundary conditions at the surface of the scatterer as simply as possible the values of m are chosen so that the scattered wave depends on ϕ in the same way as the incident wave.

The scattered electric field is then expressed explicitly as

$$\underline{E}^{s} = \frac{E_0}{k} \sum_{\ell=0}^{\infty} \left(\alpha_{\ell}\, {}^{(3)}\underline{M}^{x}_{\ell 0} + \beta_{\ell}\, {}^{(3)}\underline{M}^{z}_{\ell 1} \right) \tag{61}$$

where the ${}^{(3)}\underline{M}^{x}_{\ell 0}$ and ${}^{(3)}\underline{M}^{z}_{\ell 1}$ are inversely proportional to length, $k = 2\pi/\lambda$, and the α_{ℓ} and β_{ℓ} are arbitrary constants to be determined by the boundary conditions at the surface of the scatterer. If the superscript (i) is replaced by the superscript (3) in Equation (57) the expression for ${}^{(3)}\underline{M}^{x}_{\ell 0}$ is

$$
{}^{(3)}\underline{M}^{x}_{\ell 0} = - \underline{i}_{\eta}\, F^{-1} \sqrt{1+\xi^2}\, (\eta^2+\xi^2)^{-1/2} \sin\phi\, U_{\ell 0}(\eta) \frac{d}{d\xi}\, {}^{(3)}V_{\ell 0}(\xi)
$$

$$
+ \underline{i}_{\xi}\, F^{-1} \sqrt{1-\eta^2}\, (\eta^2+\xi^2)^{-1/2} \sin\phi\, {}^{(3)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta)
$$

$$
+ \underline{i}_{\phi}\, F^{-1} \Big[\xi (1-\eta^2)(\eta^2+\xi^2)^{-1} \cos\phi\, {}^{(3)}V_{\ell 0}(\xi) \frac{d}{d\eta} U_{\ell 0}(\eta)
$$

$$
+ \eta (1+\xi^2)(\eta^2+\xi^2)^{-1} \cos\phi\, U_{\ell 0}(\eta) \frac{d}{d\xi}\, {}^{(3)}V_{\ell 0}(\xi) \Big] . \tag{62}
$$

Also,

$$
{}^{(3)}\underline{M}^{z}_{\ell 1} = \underline{i}_{\eta}\, F^{-1} \Big[\eta (1-\eta^2)^{-1/2} (\eta^2+\xi^2)^{-1/2} \sin\phi\, U_{\ell 1}(\eta)\, {}^{(3)}V_{\ell 1}(\xi) \Big]
$$

$$
+ \underline{i}_{\xi}\, F^{-1} \Big[- \xi (1+\xi^2)^{-1/2} (\eta^2+\xi^2)^{-1/2} \sin\phi\, U_{\ell 1}(\eta)\, {}^{(3)}V_{\ell 1}(\xi) \Big]
$$

$$
+ \underline{i}_{\phi}\, F^{-1} \Big[\eta (1-\eta^2)^{1/2} (1+\xi^2)^{1/2} (\eta^2+\xi^2)^{-1} \cos\phi\, {}^{(3)}V_{\ell 1}(\xi) \frac{d}{d\eta} U_{\ell 1}(\eta)
$$

$$
- \xi (1-\eta^2)^{1/2} (1+\xi^2)^{1/2} (\eta^2+\xi^2)^{-1} \cos\phi\, U_{\ell 1}(\eta) \frac{d}{d\xi}\, {}^{(3)}V_{\ell 1}(\xi) \Big] . \tag{63}
$$

Since the total electric vector at any point is given by $\underline{E}^I + \underline{E}^S$, the boundary conditions at the surface can be expressed as

$$
\Big[E^I_{i_\eta} + E^S_{i_\eta} \Big]_{\xi = \xi_0} = 0 \;, \tag{64}
$$

$$\left[E^I_{i_\phi} + E^S_{i_\phi}\right]_{\xi = \xi_0} = 0 \quad . \tag{65}$$

Substituting the expressions for the incident and scattered fields (Eqs. 59 and 61) on the surface and using the above expressions for ${}^{(3)}\underline{M}^x_{\ell 0}$ and ${}^{(3)}\underline{M}^z_{\ell 1}$ Equations (64) and (65) become

$$i \sum_{\ell=0}^{\infty} A_{\ell 0} (1 + \xi_0^2)^{1/2} \; U_{\ell 0} (\eta) \frac{d}{d\xi} \, {}^{(1)}V_{\ell 0} (\xi_0)$$

$$= \sum_{\ell=0}^{\infty} \Big\{ -\alpha_\ell (1 + \xi_0^2)^{1/2} \; U_{\ell 0} (\eta) \frac{d}{d\xi} \, {}^{(3)}V_{\ell 0} (\xi_0)$$

$$+ \beta_\ell \, \eta (1 - \eta^2)^{-1/2} \; U_{\ell 1} (\eta) \, {}^{(3)}V_{\ell 1} (\xi_0) \Big\} \tag{66}$$

and

$$- i \sum_{\ell=0}^{\infty} A_{\ell 0} \Big[\xi_0 (1 - \eta^2) \, {}^{(1)}V_{\ell 0} (\xi_0) \frac{d}{d\eta} \; U_{\ell 0} (\eta)$$

$$+ \eta (1 + \xi_0^2) \; U_{\ell 0} (\eta) \frac{d}{d\xi} \, {}^{(1)}V_{\ell 0} (\xi_0) \Big]$$

$$= \sum_{\ell=0}^{\infty} \Big\{ \alpha_\ell \Big[\xi_0 (1 - \eta^2) \, {}^{(3)}V_{\ell 0} (\xi_0) \frac{d}{d\eta} \; U_{\ell 0} (\eta)$$

$$+ \eta (1 + \xi_0^2) \; U_{\ell 0} (\eta) \frac{d}{d\xi} \, {}^{(3)}V_{\ell 0} (\xi_0) \Big]$$

$$+ \beta_\ell \Big[\eta (1 - \eta^2)^{1/2} (1 + \xi_0^2)^{1/2} \, {}^{(3)}V_{\ell 1} (\xi_0) \frac{d}{d\eta} \; U_{\ell 1} (\eta)$$

$$- \xi_0 (1-\eta^2)^{1/2} (1+\xi_0^2)^{1/2} \; U_{\ell 1}(\eta) \frac{d}{d\xi} \, {}^{(3)}V_{\ell 1}(\xi_0) \Big] \Big\} . \tag{67}$$

The above system of equations can be solved for α_ℓ and β_ℓ by use of the orthogonal properties of the angular functions $U_{\ell m}(\eta)$. If Equations (66) and (67) are multiplied by U_{Lm} and integrated over the interval $-1 \leqq \eta \leqq 1$, then

$$i \sum_{\ell=0}^{\infty} A_{\ell 0} (1+\xi_0^2)^{1/2} \left[\frac{d}{d\xi} \, {}^{(1)}V_{\ell 0}(\xi_0) \right] I_1$$

$$= \sum_{\ell=0}^{\infty} \left\{ -\alpha_\ell (1+\xi_0^2)^{1/2} \left[\frac{d}{d\xi} \, {}^{(3)}V_{\ell 0}(\xi_0) \right] I_1 + \beta_\ell \, {}^{(3)}V_{\ell 1}(\xi_0) I_2 \right\} \tag{68}$$

and

$$-i \sum_{\ell=0}^{\infty} A_{\ell 0} \left\{ (1+\xi_0^2) \left[\frac{d}{d\xi} \, {}^{(1)}V_{\ell 0}(\xi_0) \right] I_1 + \xi_0 \, {}^{(1)}V_{\ell 0}(\xi_0) I_4 \right\}$$

$$= \sum_{\ell=0}^{\infty} \left\{ \alpha_\ell \left[(1+\xi_0^2) \left(\frac{d}{d\xi} \, {}^{(3)}V_{\ell 0}(\xi_0) \right) I_3 + \xi_0 \, {}^{(3)}V_{\ell 0}(\xi_0) I_4 \right] \right.$$

$$\left. - \beta_\ell \left[(1+\xi_0^2)^{1/2} \left(\frac{d}{d\xi} \, {}^{(3)}V_{\ell 1}(\xi_0) \right) I_5 - (1+\xi_0^2)^{1/2} \, {}^{(3)}V_{\ell 1}(\xi_0) I_6 \right] \right\} , \tag{69}$$

where

$$I_1(\ell, L) = \int_{-1}^{1} U_{\ell 0}(\eta) \, U_{L0}(\eta) \, d\eta$$

$$I_2(\ell, L) = \int_{-1}^{1} \eta (1-\eta^2)^{-1/2} \, U_{\ell 0}(\eta) \, U_{L0}(\eta) \, d\eta$$

$$I_3(\ell, L) = \int_{-1}^{1} \eta\, U_{\ell 0}(\eta)\, U_{L0}(\eta)\, d\eta$$

$$I_4(\ell, L) = \int_{-1}^{1} (1-\eta^2)\, U_{L0}(\eta) \frac{d}{d\eta} U_{\ell 0}(\eta)\, d\eta$$

$$I_5(\ell, L) = \int_{-1}^{1} (1-\eta^2)^{1/2}\, U_{\ell 1}(\eta)\, U_{L0}(\eta)\, d\eta$$

$$I_6(\ell, L) = \int_{-1}^{1} (1-\eta^2)^{1/2}\, U_{L0}(\eta) \frac{d}{d\eta} U_{\ell 1}(\eta)\, d\eta\ .$$

The six integrals above can be evaluated in terms of the Beta functions B (t, k). These functions are defined by:

$$\int_{-1}^{1} \eta^{2t}(1-\eta^2)^k\, d\eta = B(t+1/2, k+1) = \frac{2^{k+1} \qquad k!}{(2t+1)(2t+3)\cdots(2k+2t+1)} \qquad (70)$$

where k and t are non-negative integers.

For ℓ even, L even,

$$I_1(\ell, L) = I_1(E, E)^1 = \int_{-1}^{1} \left[\sum_{k=0}^{\infty} C_{2k}^{\ell 0}(1-\eta^2)^k\right]\left[\sum_{k=0}^{\infty} C_{2k}^{L0}(1-\eta^2)^k\right] d\eta$$

$$= \int_{-1}^{1} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} C_{2s}^{\ell 0}\, C_{2k-2s}^{L0}\, (1-\eta^2)^k\, d\eta$$

$$= \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} C_{2s}^{\ell 0}\, C_{2k-2s}^{L0} \int_{-1}^{1} (1-\eta^2)^k\, d\eta$$

---

[1]E and O refer to even and odd values respectively of ℓ or L.

$$= \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0} \; C_{2k-2s}^{L0} \quad B(1/2, k+1)$$

$$= \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0} \; C_{2k-2s}^{L0} \quad \frac{2^{k+1} \quad k!}{3 \cdot 5 \cdots (2k+1)} \; . \tag{71a}$$

For ℓ odd, L odd,

$$I_1 (\ell, L) = I_1 (O, O)$$

$$= \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0} \; C_{2k-2s}^{L0} \quad B(3/2, k+1)$$

$$= \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0} \; C_{2k-2s}^{L0} \quad \frac{2^{k+1} \quad k!}{3 \cdot 5 \cdots (2k+3)} \; . \tag{71b}$$

Since the angular eigenfunctions, $U_{\ell m} (\eta)$, are orthogonal in the interval, $(-1, 1)$,

$$I_1 (\ell, L) = \delta_{\ell L} \; N_{\ell 0} \tag{71c}$$

where $\delta_{\ell L}$ is the Kronecker delta function and $N_{\ell 0}$ is given in Equations (28) and (29).

Using a technique similar to that in the evaluation of I_1, the remaining integrals become:

$$I_2 (E, E) = I_2 (O, O) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 1} \; C_{2k-2s}^{L0} \quad B(3/2, k+1) \tag{72a}$$

$$I_2 (E, O) = I_2 (O, E) = 0; \tag{72b}$$

$$I_3\,(E,E) = I_3\,(O,O) = 0 \tag{73a}$$

$$I_3\,(E,O) = I_3\,(O,E) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B\,(1/2,k+1); \tag{73b}$$

$$I_4\,(E,E) = I_4\,(O,O) = 0 \tag{74a}$$

$$I_4\,(E,O) = -2 \sum_{k=0}^{\infty} \sum_{s=0}^{k} s\; C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B\,(3/2,k+1) \tag{74b}$$

$$I_4\,(O,E) = 2 \sum_{k=0}^{\infty} \sum_{s=0}^{k} (k+1-s)\; C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B\,(3/2,k+1)\,; \tag{74c}$$

$$I_5\,(E,E) = I_5\,(O,O) = 0 \tag{75a}$$

$$I_5\,(E,O) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B\,(3/2,k+1) \tag{75b}$$

$$I_5\,(O,E) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B(1/2,k+1); \tag{75c}$$

$$I_6\,(E,E) = I_6\,(O,O) = 0 \tag{76a}$$

$$I_6\,(E,O) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 1}\; C_{2k-2s}^{L0} \left[B(3/2,k+2)-(2s+1)B(5/2,k+1)\right] \tag{76b}$$

$$I_3\,(E,E) = I_3\,(O,O) = 0 \tag{73a}$$

$$I_3\,(E,O) = I_3\,(O,E) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B\,(1/2, k+1); \tag{73b}$$

$$I_4\,(E,E) = I_4\,(O,O) = 0 \tag{74a}$$

$$I_4\,(E,O) = -2 \sum_{k=0}^{\infty} \sum_{s=0}^{k} s\, C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B\,(3/2, k+1) \tag{74b}$$

$$I_4\,(O,E) = 2 \sum_{k=0}^{\infty} \sum_{s=0}^{k} (k+1-s)\, C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B\,(3/2, k+1)\,; \tag{74c}$$

$$I_5\,(E,E) = I_5\,(O,O) = 0 \tag{75a}$$

$$I_5\,(E,O) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B\,(3/2, k+1) \tag{75b}$$

$$I_5\,(O,E) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 0}\; C_{2k-2s}^{L0}\; B(1/2, k+1); \tag{75c}$$

$$I_6\,(E,E) = I_6\,(O,O) = 0 \tag{76a}$$

$$I_6\,(E,O) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 1}\; C_{2k-2s}^{L0} \left[B(3/2, k+2) - (2s+1) B(5/2, k+1) \right] \tag{76b}$$

$$I_6\,(O,E) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} C_{2s}^{\ell 1}\; C_{2k-2s}^{L0}\;(2s+1)\;B\,(3/2,\,k+1) \tag{76c}$$

To simplify the writing of Equations (68) and (69), let

$$B_{\ell}^{L} = A_{\ell 0}\,(1+\xi_0^2)^{1/2}\left[\frac{d}{d\xi}\;{}^{(1)}V_{\ell 0}\,(\xi_0)\right] I_1\,(\ell, L)$$

$$D_{\ell}^{L} = -(1+\xi_0^2)^{1/2}\left[\frac{d}{d\xi}\;{}^{(3)}V_{\ell 0}\,(\xi_0)\right] I_1\,(\ell, L)$$

$$G_{\ell}^{L} = {}^{(3)}V_{\ell 1}\,(\xi_0)\;I_2\,(\ell, L)$$

$$R_{\ell}^{L} = A_{\ell 0}\,(1+\xi_0^2)^{1/2}\left[\frac{d}{d\xi}\;{}^{(1)}V_{\ell 0}\,(\xi_0)\right] I_3\,(\ell, L) + \xi_0\;{}^{(1)}V_{\ell 0}\,(\xi_0)\;I_4(\ell, L)$$

$$S_{\ell}^{L} = (1+\xi_0^2)\left[\frac{d}{d\xi}\;{}^{(3)}V_{\ell 0}\,(\xi_0)\right] I_3\,(\ell, L) + \xi_0\;{}^{(3)}V_{\ell 0}\,(\xi_0)\;I_4\,(\ell, L)$$

$$T_{\ell}^{L} = -\,\xi_0\,(1+\xi_0^2)^{1/2}\left[\frac{d}{d\xi}\;{}^{(3)}V_{\ell 0}\,(\xi_0)\right] I_5\,(\ell, L)$$

$$+\,(1+\xi_0^2)^{1/2}\;{}^{(3)}V_{\ell 0}\,(\xi_0)\;I_6\,(\ell, L)\,. \tag{77}$$

By use of this notation, Equations (68) and (69) take the form

$$\sum_{\ell=0}^{\infty} (\alpha_{\ell}\,D_{\ell}^{L} + \beta_{\ell}\,G_{\ell}^{L}) = \sum_{\ell=0}^{\infty} B_{\ell}^{L} \quad (L = 0, 1, 2\,\ldots) \tag{78}$$

$$\sum_{\ell=0}^{\infty} (\alpha_{\ell}\,S_{\ell}^{L} + \beta_{\ell}\,T_{\ell}^{L}) = \sum_{\ell=0}^{\infty} R_{\ell}^{L} \quad (L = 0, 1, 2\,\ldots) \tag{79}$$

To obtain an approximation to the solution of these two independent sets of equations, write the first four equations of each set as:

$$\alpha_0 D_0^0 + 0 + 0 + 0 \ldots + \beta_0 G_0^0 + 0 + \beta_2 G_2^0 + 0 + \ldots = B_0^0$$

$$0 + \alpha_1 D_1^1 + 0 + 0 + \ldots + 0 + \beta_1 G_1^1 + 0 + \beta_3 G_3^1 = \ldots = B_1^1$$

$$0 + 0 + \alpha_2 D_2^2 + 0 + \ldots + \beta_0 G_0^2 + 0 + \beta_2 G_2^2 + 0 \ldots = B_2^2$$

$$0 + 0 + 0 + \alpha_3 D_3^3 + 0 + \ldots + \beta_1 G_1^3 + 0 + \beta_3 G_3^3 + \ldots = B_3^3 \qquad (80)$$

and

$$0 + \alpha_1 S_1^0 + 0 + \alpha_3 S_3^0 + 0 + \ldots + 0 + \beta_1 T_1^0 + 0 + \beta_3 T_3^0 + 0 + \ldots$$
$$= 0 + R_1^0 + 0 + R_3^0 + \ldots$$

$$\alpha_0 S_0^1 + 0 + \alpha_2 S_2^1 + 0 + \ldots + \beta_0 T_0^1 + 0 + \beta_2 T_2^1 + 0 + \ldots$$
$$= R_0^1 + 0 + R_2^1 + 0 + \ldots$$

$$0 + \alpha_1 S_1^2 + 0 + \alpha_3 S_3^2 + 0 + \ldots + 0 + \beta_1 T_1^2 + 0 + \beta_3 T_3^2 + 0 + \ldots$$
$$= 0 + R_1^2 + 0 + R_3^2 + \ldots$$

$$\alpha_0 S_0^3 + 0 + \alpha_2 S_2^3 + 0 + \ldots \beta_0 T_0^3 + 0 + \beta_2 T_2^3 + 0 + \ldots$$
$$= R_0^3 + 0 + R_2^3 + 0 + \ldots \qquad (81)$$

Here α_{2n} and β_{2n} $(n = 0, 1, \ldots)$ occur in the first, third, ... equations of system (80) and in the second, fourth, ... equations of system (81). In particular, as a first approximation, to α_0 and β_0, consider only the first equations containing α_0 and β_0 in each set. Then

$$\alpha_0 = \frac{\begin{vmatrix} B_0^0 & G_0^0 \\ R_0^1 & T_0^1 \end{vmatrix}}{\begin{vmatrix} D_0^0 & G_0^0 \\ S_0^1 & T_0^1 \end{vmatrix}} \qquad (82)$$

and

$$\beta_0 = \frac{\begin{vmatrix} D_0^0 & B_0^0 \\ S_0^1 & R_0^1 \end{vmatrix}}{\begin{vmatrix} D_0^0 & G_0^0 \\ S_0^1 & T_0^1 \end{vmatrix}} . \qquad (83)$$

The α_{2n+1} and β_{2n+1} $(n = 0, 1, \ldots)$ occur in the second, fourth, ... equations of system (80) and in the first, third, ... equations of system (81). Hence, by a similar process for $n = 0$,

$$\alpha_1 = \frac{\begin{vmatrix} B_1^1 & G_1^1 \\ R_1^0 & T_1^0 \end{vmatrix}}{\begin{vmatrix} D_1^1 & G_1^1 \\ S_1^0 & T_1^0 \end{vmatrix}} \qquad (84)$$

and

$$\beta_1 = \frac{\begin{vmatrix} D_1^1 & B_1^1 \\ S_0^1 & R_1^0 \end{vmatrix}}{\begin{vmatrix} D_1^1 & G_1^1 \\ S_1^0 & T_1^0 \end{vmatrix}} . \tag{85}$$

For a general α_{2n} or β_{2n} $(n = 1, 2, \ldots)$ the $n + 1$ equations from each system in which α_{2n} and β_{2n} appear must be used. Similarly, for a general α_{2n+1} or β_{2n+1} $(n = 1, 2, \ldots)$ $n + 1$ equations from each system in which α_{2n+1} and β_{2n+1} appear must be used.

Thus in the above expressions for α_n and β_n the quantities B_{ℓ}^L, D_{ℓ}^L, G_{ℓ}^L, R_{ℓ}^L, S_{ℓ}^L, and T_{ℓ}^L each depend upon the oblate spheroidal coefficients $C_{2k}^{\ell m}$. These spheroidal coefficients are completely determined by the defining equations for $C_0^{\ell m}$ and $C_2^{\ell m}$ and the recursion formulas (16) and (17).

V

ASYMPTOTIC FORM OF THE SCATTERED ELECTRIC FIELD

In the previous chapter an expression was obtained for the scattered electromagnetic field everywhere in space. The back-scattering cross-section, however, depends explicitly only on the nature of the electromagnetic field in the vicinity of the source, that is, in the asymptotic behavior of the scattered field as $\epsilon \to \infty$ and $\eta = 1$.

The expansion of $\underline{E}^s$ was given in the form

$$\underline{E}^s = \frac{E_0}{k} \sum_{\ell=0}^{\infty} \left(a_\ell \, {}^{(3)}\underline{M}^x_{\ell 0} + \beta_\ell \, {}^{(3)}\underline{M}^z_{\ell 1} \right) \tag{61}$$

where ${}^{(3)}\underline{M}^x_{\ell 0}$ is given by Equation (62) and ${}^{(3)}\underline{M}^z_{\ell 1}$ by Equation (63).

Substituting the asymptotic forms of ${}^{(3)}V_{\ell m}$, [Eqs. (36) and (37)] into Equations (62) and (63), and letting $\eta = 1$, leads to

$${}^{(3)}\underline{M}^x_{\ell 0} = \left[-\underline{i}_\eta \sin \phi + \underline{i}_\phi \cos \phi \right] q_{\ell 0} \frac{e^{ikr}}{r} \tag{86}$$

and

$${}^{(3)}\underline{M}^z_{\ell 1} = 2 \epsilon^{-1} k^{-1} \left[-\underline{i}_\eta \sin \phi + \underline{i}_\phi \cos \phi \right] q_{\ell 1} \frac{e^{ikr}}{r^2} . \tag{87}$$

For $\eta = 1$, Equations (3a) and (3c) reduce to

$$\underline{i}_\eta = -\underline{i}_x \cos \phi - \underline{i}_y \sin \phi \tag{88}$$

and

$$\underline{i}_\phi = -\underline{i}_x \sin \phi + \underline{i}_y \cos \phi . \tag{89}$$

So that, Equations (86) and (87) become

$$^{(3)}\underline{M}^{x}_{\ell 0} = \underline{i}_y \; a_{\ell 0} \; \frac{e^{ikr}}{r} \tag{90}$$

and

$$^{(3)}\underline{M}^{z}_{\ell 1} = 2 \, \underline{i}_y \; q_{\ell 1} \; \epsilon^{-1} \; k^{-1} \; \frac{e^{ikr}}{r^2} \; . \tag{91}$$

Substituting Equations (90) and (91) into Equation (61) and neglecting term of order $1/r^2$ the asymptotic form of the scattered electric field is given by

$$\underline{E}^s = \underline{i}_y \; \frac{E_0}{k} \; \frac{e^{ikr}}{r} \sum_{\ell=0}^{\infty} a_\ell \; q_{\ell 0} \; . \tag{92}$$

VI

THE BACK-SCATTERING CROSS-SECTION OF THE OBLATE SPHEROID

The asymptotic expression for the scattered electric field (Eq. 92) makes it possible to formulate the far zone back-scattering cross-section explicitly.

The scattering cross-section, σ, is defined as the cross-section of an isotropic scatterer which would scatter in the direction of interest the same amount of energy as the oblate spheroid scatters in that direction. If P^I denotes the total power intercepted by an isotropic scatterer of cross-section σ, then

$$P^I = \sigma \left| \underline{P}^I \right| = \sigma \left| \underline{E}^I \times \underline{H}^I \right| = \sigma \sqrt{\frac{\epsilon_0}{\mu_0}} \, (E_0)^2 \tag{93}$$

where $\underline{P}^I$ is the Poynting vector of the incident wave.

Thus the magnitude of the Poynting vector for the wave scattered by the isotropic scatterer is

$$\left| \underline{P}^S \right| = \frac{P^I}{4\pi r^2} = \frac{1}{4\pi} \sqrt{\frac{\epsilon_0}{\mu_0}} \, \frac{E_0^2}{r^2} \, \sigma \tag{94}$$

where r is the distance from the center of the isotropic scatterer to the point of observation.

But $\left| \underline{P}^S \right|$ is also given by

$$\left| \underline{P}^S \right| = \left| \underline{E}^S \times \underline{H}^S \right| = \sqrt{\frac{\epsilon_0}{\mu_0}} \left| \underline{E}^S \right|^2 = \sqrt{\frac{\epsilon_0}{\mu_0}} \, \frac{E_0^2}{k^2 r^2} \left| \sum_{\ell=0}^{\infty} a_\ell \, q_{\ell 0} \right|^2 . \tag{95}$$

So that, combining Equations (94) and (95), σ is expressed in terms of the constants a_ℓ of the scattered electric field as

$$\sigma = \frac{4\pi}{k^2}\left|\sum_{\ell=0}^{\infty} a_\ell q_{\ell 0}\right|^2 = \frac{\lambda^2}{\pi}\left|\sum_{\ell=0}^{\infty} a_\ell \; q_{\ell 0}\right|^2 , \tag{96}$$

or

$$\sigma = \frac{4\pi F^2}{\epsilon^2}\left|\sum_{\ell=0}^{\infty} a_\ell \; q_{\ell 0}\right|^2 \tag{96a}$$

where $q_{\ell 0}$ is given in Equations (34) and (35). F is one half the focal length and $\epsilon = \frac{2\pi F}{\lambda}$, λ being the wavelength of the incident radiation.

Leitner and Spence (Ref. 3) have tabulated $q_{\ell 0}$ for ϵ = 1, 2, 3, 4, and 5 and for ℓ = 0, 1, ... 5. This tabulation of $q_{\ell 0}$ must be extended to larger values of ℓ and ϵ if numerical values of σ are to be obtained.

APPENDIX I

SCATTERING FROM A CIRCULAR DISK

In the limit, $\xi_0 \to 0$, the oblate spheroid approaches a circular disk of radius F. Bouwkamp (Ref. 9) has shown that the scattered field from such a disk will be infinite at its edge. Flammer (Ref. 10) has solved the disk problem in terms of oblate spheroidal wave functions which possess the correct singularities. To demonstrate that the scattered wave has the correct singularity at $\xi = 0$, the expansion of the radial functions in terms of Hankel functions used in this paper is inappropriate and Equations (34) and (46) of Reference 3, giving the functions as power series in ξ, should be used.

Then it can be shown that Equation (6) of Part II in Reference 10 and Equation (61) of this report differ essentially only in the presence of even values of ℓ in the summation of Equation (61). However, even values of ℓ imply even values of ℓ-m in ${}^{(3)}M^{x}_{\ell 0}$ and odd values of ℓ-m in ${}^{(3)}M^{z}_{\ell 1}$. It can be shown that the radial functions of the first kind vanish at $\xi = 0$, for ℓ-m odd, while their derivatives vanish at $\xi = 0$ for ℓ-m even. This implies that $B^L_\ell = R^L_\ell = 0$ for ℓ even, in Equation (77). Thus Equations (80) for the α_ℓ and β_ℓ with ℓ even are homogeneous equations in the disk limit, with the solutions

$$\alpha_0 = \alpha_2 = \alpha_4 = \ldots = \beta_0 = \beta_2 = \beta_4 = \ldots = 0 .$$

Hence (61) does in fact reduce to an odd sum over ℓ. The singularities are discussed in detail in Reference 10.

The edge condition of vanishing tangential electric field, which was used by Flammer to obtain a unique solution, follows automatically from the solution presented here which required $E_\phi = 0$ for all values of ξ_0.

APPENDIX II

THE SCATTERING CROSS-SECTION OF THE PROLATE SPHEROID

The oblate spheroid can be transformed into the prolate spheroid by the simple transformation:

$$\begin{aligned}\eta \text{ oblate} &= \eta \text{ prolate},\\ \xi \text{ oblate} &= i\xi \text{ prolate},\\ \phi \text{ oblate} &= \pi-\phi \text{ prolate, and}\\ \epsilon \text{ oblate} &= i\epsilon \text{ prolate}.\end{aligned} \tag{A}$$

Therefore, if the oblate spheroidal functions and oblate spheroidal coefficients are transformed into prolate spheroidal functions and prolate spheroidal coefficients respectively, the solution of the boundary value problem for the oblate spheroid should be transformed into the solution of the same boundary value problem for the prolate spheroid.

This transformation of solutions does, in fact, occur. When the coordinate transformation is applied, the oblate spheroid becomes a prolate spheroid and the separated angular and radial differential equations are arrived at from Equation (12). These equations are satisfied by the prolate spheroidal functions. Furthermore, the recursion formulas for the coefficients $C_{2k}^{\ell m}$ take the form,

$$2k(2k+2m)\,C_{2k}^{\ell m} - \left[(2k-\ell+m-2)(2k+\ell+m-1) - \alpha_{\ell m}\right] C_{2k}^{\ell m} + \epsilon^2 \, C_{2k-4}^{\ell m} = 0 \quad (\ell - m) \text{ even} \tag{B}$$

and

$$2k\,(2k + 2m)\, C_{2k}^{\ell m} - \left[(2k - \ell + m - 1)(2k + \ell + m) - a_{\ell m}\right] C_{2k}^{\ell m}$$

$$+ \epsilon^2 \, C_{2k-4}^{\ell m} = 0 \quad (\ell - m) \text{ odd} \quad . \qquad \text{(C)}$$

Because these recursion formulas differ for the oblate and prolate spheroidal systems, the spheroidal coefficients are different, numerically, in the two systems. This numerical difference leads to different values of the scattered electromagnetic field and of the scattering cross-section even though the explicit form of these two quantities is the same in both coordinate systems.

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